AN INVOLUTION ON $\beta(1,0)$-TREES

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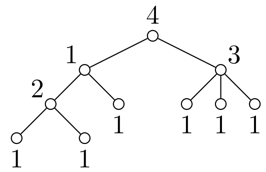
Abstract. In [Decompositions and statistics for $\beta(1,0)$-trees and nonseparable permutations, Advances Appl. Math. 42 (2009) 313–328] we introduced an involution, h, on $\beta(1,0)$-trees. We neglected, however, to prove that h indeed is an involution. In this note we provide the missing proof. We also refine an equidistribution result given in the same paper.

1. Introduction

A $\beta(1,0)$-*tree* [2] is a rooted plane tree labeled with positive integers such that

(1) Leaves have label 1.
(2) The root has label equal to the sum of its children's labels.
(3) Any other node has label no greater than the sum of its children's labels.

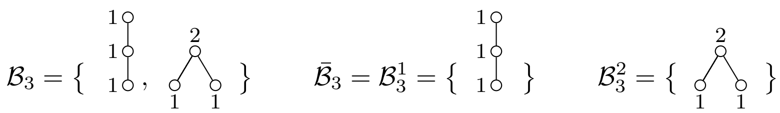
Below is an example of such a tree.



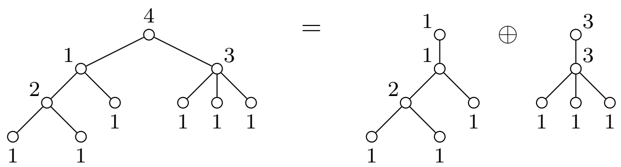
In [1] we introduced an involution, h, on $\beta(1,0)$-trees. We also gave a result on the equidistribution of certain statistics on $\beta(1,0)$-trees. A proof that h indeed is an involution was, however, not given; rather, the proof was said to be found in a forthcoming paper that never materialised. The proof of the equidistribution was in fact also omitted. In this note we give the two missing proofs. We also refine the equidistribution result.

2. The structure of $\beta(1,0)$-trees

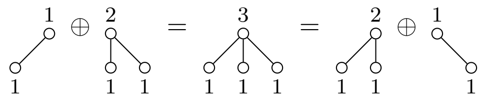
We say a $\beta(1,0)$-tree on two or more nodes is *indecomposable* if its root has exactly one child and *decomposable* if it has more than one child. The $\beta(1,0)$-tree on one node, $\circ = \circ\, 1$, is neither indecomposable nor decomposable. Let $\mathcal{B}_n$ be the set of all $\beta(1,0)$-trees on n nodes, and let $\bar{\mathcal{B}}_n$ be the subset of $\mathcal{B}_n$ consisting of the indecomposable trees. Let $\mathcal{B}_n^k$ be the subset of $\mathcal{B}_n$ consisting of the trees with root label k. For instance,



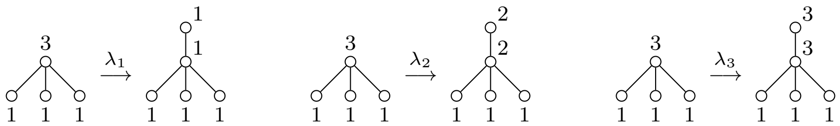
Decomposable trees can be regarded as sums of indecomposable ones:



In fact we do not need to require u and v to be indecomposable for the sum $u \oplus v$ to make sense. In general, we define that the root label of $u \oplus v$ is the sum of the root label of u and the root label of v, and that the subtrees of $u \oplus v$ are those of u followed by those of v. So,



Further, there is a simple one-to-one correspondence λ between the Cartesian product $[k] \times \mathcal{B}_{n-1}^k$ and the disjoint union $\cup_{i=1}^k \bar{\mathcal{B}}_n^i$, where $\bar{\mathcal{B}}_n^k$ is the subset of $\bar{\mathcal{B}}_n$ consisting of the trees with root label k:

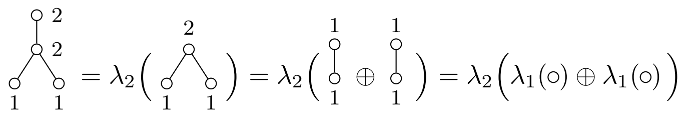


In general, if t is a tree with root label k and i is an integer such that $1 \le i \le k$, then $\lambda_i t$ is obtained from t by joining a new root via an edge to the old root; and both the new root and the old root are assigned the label i.

Thus each $\beta(1,0)$-tree, t, is of exactly one the following three forms:

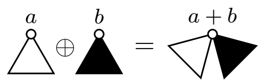
$t = \circ$, (the single node tree)
$t = u \oplus v$, (decomposable)
$t = \lambda_i u$, where $1 \le i \le \operatorname{root} u$, (indecomposable)

in which u and v are $\beta(1,0)$-trees, and $\operatorname{root} u$ denotes the root label of u. As an example of the encoding this characterisation entails we have

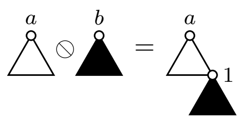


3. An involution on $\beta(1,0)$-trees

In this section we define an involution on $\beta(1,0)$-trees. To that end we now describe a new way of decomposing $\beta(1,0)$-trees. Schematically the sum $\oplus$ on $\beta(1,0)$-trees is described by



An alternative sum is



That is, to get $u \oslash v$ we join u and v by identifying the rightmost leaf in u with the root of v, and that node is assigned the label 1.

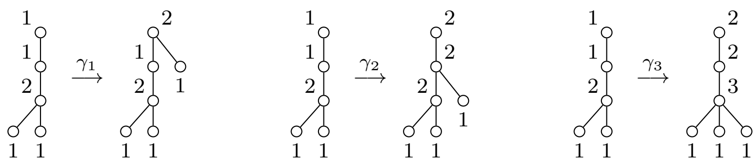
The *right path* is the path from the root to the rightmost leaf. Let $\operatorname{rpath}(t)$ denote the length of (number of edges on) the right path of t. Note that

$$\operatorname{root}(u \oplus v) = \operatorname{root} u + \operatorname{root} v \tag{1}$$
$$\operatorname{rpath}(u \oplus v) = \operatorname{rpath} v \tag{2}$$

while

$$\operatorname{root}(u \oslash v) = \operatorname{root} u \tag{3}$$
$$\operatorname{rpath}(u \oslash v) = \operatorname{rpath} u + \operatorname{rpath} v. \tag{4}$$

for $u \neq \circ$ and $v \neq \circ$. Thus, with respect to $\obslash$, rpath plays the role of root, and vice versa. There is also a map γ that plays a role analogous to that of λ:

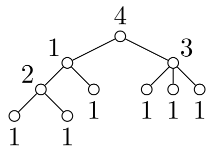


Here is how $\gamma_i t$ is defined in general: Assume that the length of the right path of t is k and that i is an integer such that $1 \leq i \leq k$. Let us by x refer to the ith node on the right path of t. Then $\gamma_i t$ is obtained from t by joining a new leaf via an edge to x, making the new leaf the rightmost leaf in $\gamma_i t$; and, lastly, add 1 to the label of each node on the right path, except for the new leaf. Note that $\operatorname{rpath} \gamma_i t = i$.

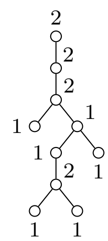
We explore the two ways to decompose $\beta(1,0)$-trees we now have by defining an endofunction $h : \mathcal{B} \to \mathcal{B}$ as follows:

$$h(\circ) = \circ;$$
$$h(\lambda_i t) = \gamma_i h(t);$$
$$h(u \oplus v) = h(v) \obslash h(u).$$

For instance,



$$= \lambda_1\Big(\lambda_2\big(\substack{\circ\\\circ} \oplus \substack{\circ\\\circ}\big) \oplus \substack{\circ\\\circ}\Big) \oplus \lambda_3\big(\substack{\circ\\\circ} \oplus \substack{\circ\\\circ} \oplus \substack{\circ\\\circ}\big)$$
$$\xrightarrow{h} \gamma_3\big(\substack{\circ\\\circ} \obslash \substack{\circ\\\circ} \obslash \substack{\circ\\\circ}\big) \obslash \gamma_1\Big(\substack{\circ\\\circ} \obslash \gamma_2\big(\substack{\circ\\\circ} \obslash \substack{\circ\\\circ}\big)\Big) =$$



We will soon see that h is in fact an involution! First we state some almost self-evident lemmas about relations between $\oplus$, $\obslash$, λ, and γ.

Lemma 1. *Let t, u, and v be $\beta(1,0)$-trees. Then*

$$t \oplus (u \obslash v) = (t \oplus u) \obslash v.$$

Lemma 2. *Let u and v be $\beta(1,0)$-trees. Then*

$$\lambda_i(u \obslash v) = (\lambda_i u) \obslash v;$$
$$\gamma_i(u \oplus v) = u \oplus (\gamma_i v).$$

Lemma 3. *Let t be a $\beta(1,0)$-tree. Then*

$$\gamma_1 t = t \oplus \substack{\circ\\\circ};$$
$$\lambda_1 t = \substack{\circ\\\circ} \obslash t;$$
$$\gamma_{i+1}\lambda_j = \lambda_{j+1}\gamma_i.$$

Next we apply the lemmas above to prove the following lemma which is the most crucial component in establishing that h is an involution.

Lemma 4. *Let t, u, and v be $\beta(1,0)$-trees. Then*

$$h(\circ) = \circ, \quad h(\gamma_i t) = \lambda_i h(t), \quad \textit{and} \quad h(u \obslash v) = h(v) \oplus h(u).$$

Proof. We use induction on the number of nodes. The base case is trivial. The proof of the second claim is split into two cases:
Case 1, $t = \lambda_j u$: We shall prove that $h(\gamma_i\lambda_j u) = \lambda_i h(\lambda_j u)$ for all positive integers i and j. If $i = 1$, then

$$\begin{aligned}
h(\gamma_1\lambda_j u) &= h(\lambda_j u \oplus \begin{smallmatrix}\circ\\|\\\circ\end{smallmatrix}) && \text{by Lemma 3}\\
&= h(\begin{smallmatrix}\circ\\|\\\circ\end{smallmatrix}) \oslash h(\lambda_j u) && \text{by definition of } h\\
&= \begin{smallmatrix}\circ\\|\\\circ\end{smallmatrix} \oslash \gamma_j h(u) && \text{by definition of } h\\
&= \lambda_1\gamma_j h(u) && \text{by Lemma 3}\\
&= \lambda_1 h(\lambda_j u) && \text{by definition of } h
\end{aligned}$$

If $i > 1$, then

$$\begin{aligned}
h(\gamma_i\lambda_j u) &= h(\lambda_{j+1}\gamma_{i-1}u) && \text{by Lemma 3}\\
&= \gamma_{j+1}h(\gamma_{i-1}u) && \text{by definition of } h\\
&= \gamma_{j+1}\lambda_{i-1}h(u) && \text{by induction}\\
&= \lambda_i\gamma_j h(u) && \text{by Lemma 3}\\
&= \lambda_i h(\lambda_j u) && \text{by definition of } h
\end{aligned}$$

Case 2, $t = u \oplus v$:

$$\begin{aligned}
h\gamma_i(u \oplus v) &= h(u \oplus \gamma_i v) && \text{by Lemma 2}\\
&= h(\gamma_i v) \oslash h(u) && \text{by definition of } h\\
&= \lambda_i h(v) \oslash h(u) && \text{by induction}\\
&= \lambda_i\big(h(v) \oslash h(u)\big) && \text{by Lemma 2}\\
&= h(u \oplus v) && \text{by definition of } h
\end{aligned}$$

The proof of the third claim is also split into two cases.
Case 1, $u = \lambda_i t$:

$$\begin{aligned}
h(\lambda_i t \oslash v) &= h\lambda_i(t \oslash v) && \text{by Lemma 2}\\
&= \gamma_i h(t \oslash v) && \text{by Lemma 4}\\
&= \gamma_i\big(h(t) \oplus h(v)\big) && \text{by induction}\\
&= h(v) \oplus \gamma_i h(t) && \text{by Lemma 2}\\
&= h(v) \oplus h(\lambda_i t) && \text{by definition of } h
\end{aligned}$$

Case 2, $u = s \oplus t$:

$$\begin{aligned}
h((s \oplus t) \oslash v) &= h(s \oplus (t \oslash v)) && \text{by Lemma 1}\\
&= h(t \oslash v) \oslash h(s) && \text{by definition of } h\\
&= \big(h(v) \oplus h(t)\big) \oslash h(s) && \text{by induction}\\
&= h(v) \oplus \big(h(t) \oslash h(s)\big) && \text{by Lemma 1}\\
&= h(v) \oplus h\big(s \oplus t\big) && \text{by definition of } h
\end{aligned}$$

□

Theorem 5. *The function h is an involution.*

Proof. We proceed by induction. By definition $h(\circ) = \circ$; using that twice the base case follows. For the induction step we consider indecomposable and decomposable trees separately. First, for indecomposable trees:

$$h^2(\lambda_i t) = h\big(\gamma_i h(t)\big) = \lambda_i h^2(t) = \lambda_i(t).$$

Here we have used the definition of h, Lemma 4, and the induction hypothesis. Second, for decomposable trees:

$$h^2(u \oplus v) = h\bigl(h(v) \oslash h(u)\bigr) = h^2(v) \oplus h^2(u) = v \oplus u.$$

Again, we used the definition of h, Lemma 4, and the induction hypothesis. $\square$

4. Statistics on $\beta(1,0)$-trees

Let t be a $\beta(1,0)$-tree. Recall that by $\operatorname{root} t$ we denote the label of the root. Recall also that the *right path* is the path from the root to the rightmost leaf, and that the length of the right path is denoted $\operatorname{rpath} t$.

By $\operatorname{leaves} t$ we denote the number of leaves in t; by $\operatorname{int} t$ we denote the number of internal nodes (or nonleaves) in t. Note that the root is an internal node.

The number of subtrees (or, equivalently, the number of children of the root) is denoted $\operatorname{sub} t$. Further, the number of 1's below the root on the right path is denoted $\operatorname{rsub} t$.

Theorem 6. *On $\beta(1,0)$-trees with at least one edge, the involution h sends the first tuple below to the second.*

$$\begin{array}{lllllll} (& \text{leaves}, & \text{int}, & \text{root}, & \text{rpath}, & \text{sub}, & \text{rsub} &) \\ (& \text{int}, & \text{leaves}, & \text{rpath}, & \text{root}, & \text{rsub}, & \text{sub} &) \end{array}$$

Proof. We shall use induction to show that $\operatorname{rpath} h(t) = \operatorname{root} t$ and that $\operatorname{root} h(t) = \operatorname{rpath} t$; the other claims follow similarly. The base case is trivial. Assume that $t = \lambda_i u$ is indecomposable. Then

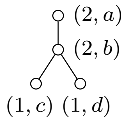
$$\operatorname{rpath} h(\lambda_i u) = \operatorname{rpath} \gamma_i h(u) = i = \operatorname{root} \lambda_i u$$

by definition of h, definition of root and rpath, respectively. Furthermore, for a decomposable tree $t = u \oplus v$ we have

$$\begin{aligned} \operatorname{rpath} h(u \oplus v) &= \operatorname{rpath}\bigl(h(u) \oslash h(v)\bigr) && \text{by definition of } h \\ &= \operatorname{rpath} h(u) + \operatorname{rpath} h(v) && \text{by (4)} \\ &= \operatorname{root} u + \operatorname{root} v && \text{by induction} \\ &= \operatorname{root}(u \oplus v) && \text{by definition of root} \end{aligned}$$

We have thus shown that $\operatorname{rpath} h(t) = \operatorname{root} t$ for any $\beta(1,0)$-tree t. That $\operatorname{root} h(t) = \operatorname{rpath} t$ follows from h being an involution. $\square$

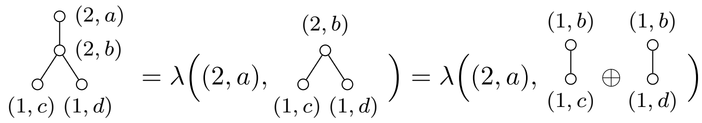
The above theorem can be strengthened by introducing what we call labeled $\beta(1,0)$-trees.



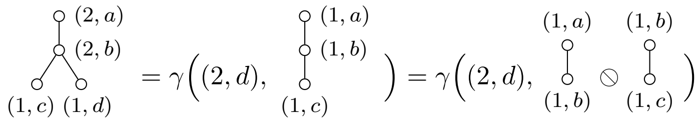
This is a $\beta(1,0)$-tree in which each node has been assigned a unique label. In this example, the labels are taken from the alphabet $\{a,b,c,d\}$. A recursive characterization of labeled $\beta(1,0)$-trees reads as follows. A *labeled $\beta(1,0)$-tree* is of exactly one of the three forms:

(0) $(1,x)$, a leaf with label x;
(1) $\lambda((i,x),t)$, where t is a labeled $\beta(1,0)$-tree and $i \leq \operatorname{root} t$;
(2) $u \oplus v$, where u and v are labeled $\beta(1,0)$-trees.

Here we assume that the function root is extended to labeled $\beta(1,0)$-trees by simply ignoring the extra labels. Also, λ and $\oplus$ are extended to labeled $\beta(1,0)$-trees in the obvious way:



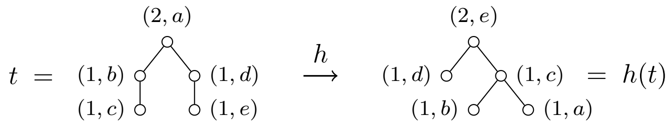
Similarly, we extend γ and $\oslash$:



The involution h is also easy to extend to $\beta(1,0)$-trees:

$$h(1,x) = (1,x);$$
$$h\lambda((i,x),t) = \gamma((i,x),h(t));$$
$$h(u \oplus v) = h(v) \oslash h(u).$$

For instance,



Let $\mathtt{v}(t)$ be the word obtained from preorder traversal of t. Also, denote by w^r the reverse of the word w. For instance, with t as above, we have $\mathtt{v}(t) = abcde$ and $\mathtt{v}(t)^r = edcba$.

Let $\mathtt{leaves}\,t$ be the subword of $\mathtt{v}(t)$ whose letters are labels of leaves of t, and let $\mathtt{int}\,t$ be the subword of $\mathtt{v}(t)^r$ whose letters are labels of internal nodes of t.

Any labeled $\beta(1,0)$-tree can be written uniquely as a sum of indecomposable $\beta(1,0)$-trees. If $t = \lambda((i_1,x_1),t_1) \oplus \cdots \oplus \lambda((i_k,x_k),t_k)$ is so written, we let $\mathtt{sub}\,t = (\,\mathtt{v}(t_1),\ldots,\mathtt{v}(t_k)\,)$. Similarly, assuming that $t = \gamma((i_1,x_1),t_1) \oslash \cdots \oslash \gamma((i_k,x_k),t_k)$ we let $\mathtt{rsub}\,t = (\,\mathtt{v}(t_k)^r,\ldots,\mathtt{v}(t_1)^r\,)$.

The definition of the statistic $\mathtt{root}\,t$ is a bit involved: $\mathtt{root}\,t$ is a subword of $\mathtt{leaves}\,t$ of length $k = \mathrm{root}\,t$. More precisely, we build this word by starting at the root and greedily and recursively searching for k leaves in its subtrees starting from the rightmost subtrees; also, we never search for more leaves in a subtree than the root label of that subtree. A more precise and formal definition can be found in the proof of Theorem 7. Let $\mathtt{rpath}\,t$ be the subword of $\mathtt{v}(v)^r$ whose letters are labels of the right path of t, excluding the leaf.

With t and $h(t)$ as in the above picture we have

$$\begin{array}{lclcl}
\mathtt{leaves}\,t & = & \mathtt{int}\,h(t) & = & ce\\
\mathtt{int}\,t & = & \mathtt{leaves}\,h(t) & = & dba\\
\mathtt{root}\,t & = & \mathtt{rpath}\,h(t) & = & ce\\
\mathtt{rpath}\,t & = & \mathtt{root}\,h(t) & = & da\\
\mathtt{sub}\,t & = & \mathtt{rsub}\,h(t) & = & (bc, de)\\
\mathtt{rsub}\,t & = & \mathtt{sub}\,h(t) & = & (d, cba).
\end{array}$$

Theorem 7. *On labeled $\beta(1,0)$-trees with at least one edge, the involution h sends the first tuple below to the second.*

$$\begin{array}{lllllll}
(& \mathtt{leaves}, & \mathtt{int}, & \mathtt{root}, & \mathtt{rpath}, & \mathtt{sub}, & \mathtt{rsub}\ )\\
(& \mathtt{int}, & \mathtt{leaves}, & \mathtt{rpath}, & \mathtt{root}, & \mathtt{rsub}, & \mathtt{sub}\ )
\end{array}$$

Proof. In terms of the recursive decomposition of labeled $\beta(1,0)$-trees, we have

$$
\begin{array}{lcl}
\texttt{leaves}\ (1,x) & = & x \\
\texttt{leaves}\,\lambda((i,x),t) & = & \texttt{leaves}\,t \\
\texttt{leaves}(u \oplus v) & = & \texttt{leaves}\,u \sqcup \texttt{leaves}\,v \\[1ex]
\texttt{int}\ (1,x) & = & x \\
\texttt{int}\,\gamma((i,x),t) & = & \texttt{int}\,t \\
\texttt{int}(u \oslash v) & = & \texttt{int}\,v \sqcup \texttt{int}\,u \\[1ex]
\texttt{root}\ (1,x) & = & x \\
\texttt{root}\,\lambda((i,x),t) & = & \mathrm{take}_i(\texttt{root}\,t) \\
\texttt{root}(u \oplus v) & = & \texttt{root}\,u \sqcup \texttt{root}\,v \\[1ex]
\texttt{rpath}\ (1,x) & = & x \\
\texttt{rpath}\,\gamma((i,x),t) & = & \mathrm{take}_i(\texttt{rpath}\,t) \\
\texttt{rpath}(u \oslash v) & = & \texttt{rpath}\,v \sqcup \texttt{rpath}\,u \\[1ex]
\texttt{sub}\ (1,x) & = & \epsilon \\
\texttt{sub}\,\lambda((i,x),t) & = & (\texttt{v}(t)) \\
\texttt{sub}(u \oplus v) & = & \texttt{sub}\,u \sqcup \texttt{sub}\,v \\[1ex]
\texttt{rsub}\ (1,x) & = & \epsilon \\
\texttt{rsub}\,\gamma((i,x),t) & = & (\texttt{v}(t)^r) \\
\texttt{rsub}(u \oslash v) & = & \texttt{rsub}\,v \sqcup \texttt{rsub}\,u
\end{array}
$$

where $u \sqcup v$ denotes the concatenation of u and v, and $\mathrm{take}_i(a_1 \dots a_n) = a_1 \dots a_i$. Using induction and the definition of h the result readily follows. □

References

[1] A. Claesson, S. Kitaev and E. Steingrímsson, Decompositions and statistics for $\beta(1,0)$-trees and nonseparable permutations, *Advances Appl. Math.* **42** (2009) 313–328.

[2] B. Jacquard, G. Schaeffer, A bijective census of nonseparable planar maps, *J. Combin. Theory Ser. A* **83**(1) (1998) 1–20.